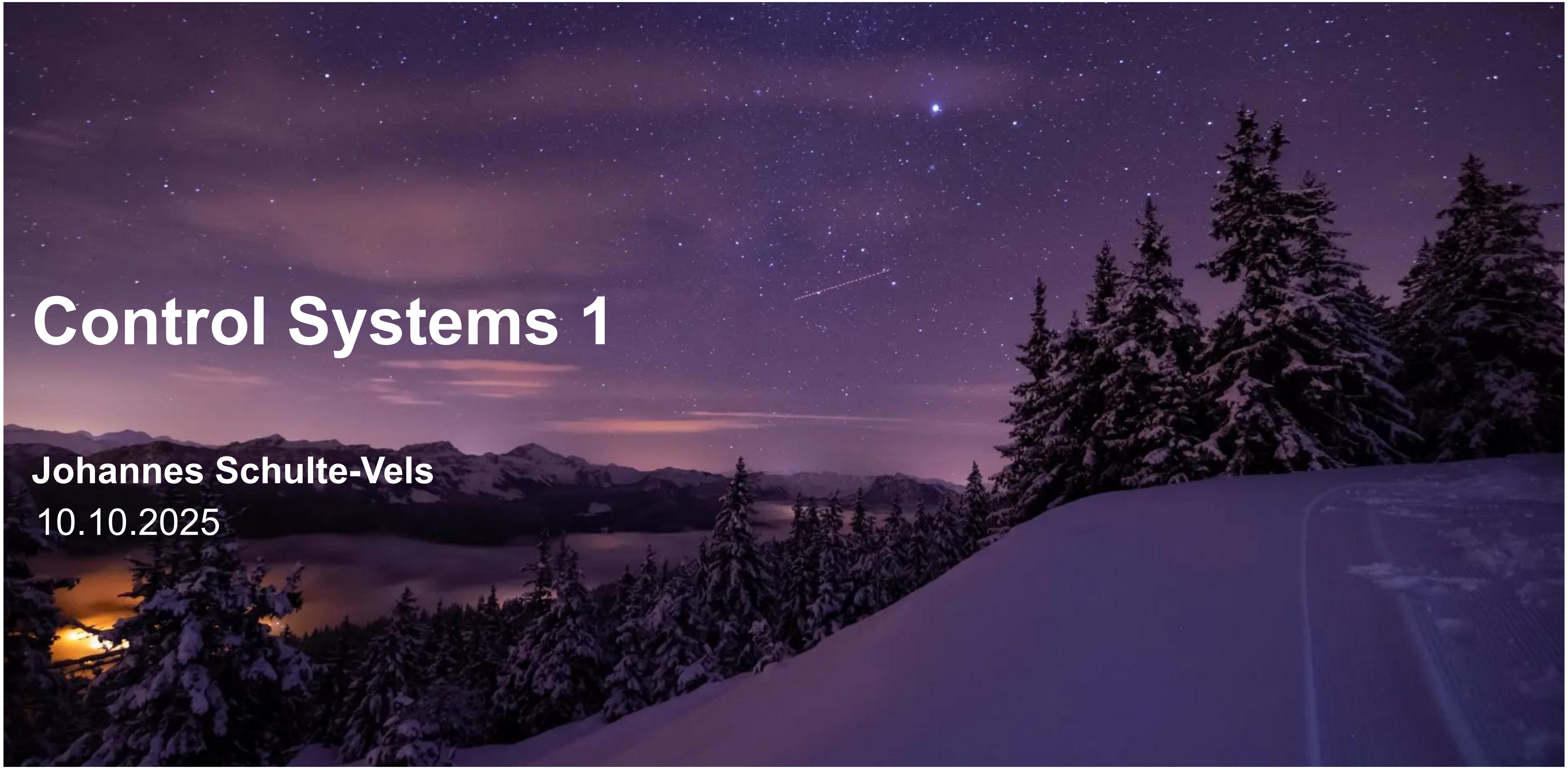
ETH zürich

Institute for Dynamic Systems and Control



Control Systems 1

Johannes Schulte-Vels

10.10.2025

Welcome!

Polybox

PW: jschul

Website

https://n.ethz.ch/~jschul

Today:

- Repetition Session 3
- Theory Recap
  - Time Response
  - Similarity Transform
  - Stability
- Q&A Session / Done

Repetition Session 3

When is a system considered to be causal?

A) Input only depends on past

C) Input only depends on present

B) Input only depends and past and present

D) Input only depends on future

Linear or Nonlinear

For a System Σ to be linear, we need to have the following properties:

1. Additivity: $\Sigma(u_1 + u_2) = \Sigma u_1 + \Sigma u_2$

2. Homogeneity: $\Sigma(\alpha u) = \alpha \Sigma u \ , \ \alpha \in \mathbb{R}$

Summarized, this leads to the following:

$$\Sigma(\alpha u_1 + \beta u_2) = \alpha \Sigma u_1 + \beta \Sigma u_2, \quad \alpha, \beta \in \mathbb{R}.$$

Causal or Noncausal

A system is called casual, **iff** (if and only if) the output depends only on past and current inputs, but never on future inputs (future does not change the present).

Only causal systems are physically realizable

Static (memoryless) or Dynamic

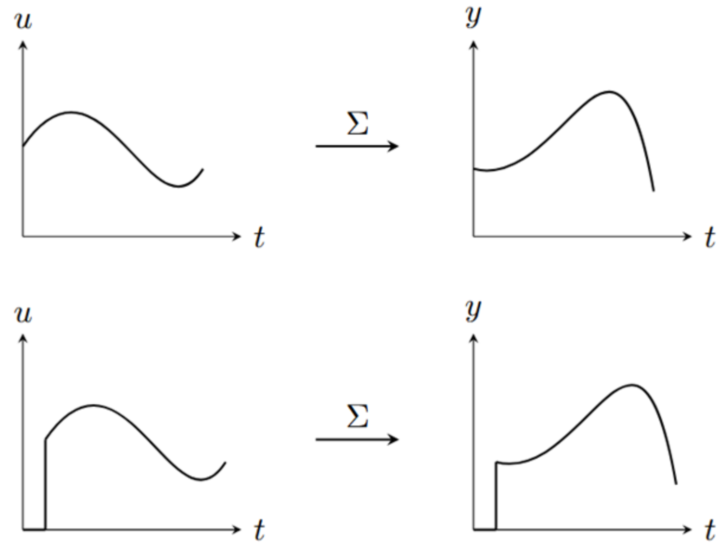
In static systems, the output only depends on the current input (no past, no future)

Time invariant or Time variant

A time invariant system will always have the same output to a certain input, independent of when the input is applied. We can shift the input in time and the output will also be shifted by the same amount.

General rule of thumb:The system is time-invariant if the system equations do not contain time t as a summand, factor or exponent. Time t only appears in u(t).

Mathematically: $y(t-\tau) = (\Sigma\tilde{u})(t), \text{ where } \tilde{u}(t) = u(t-\tau).$



HS 2023

$$\Sigma_2 : \quad y(t) = \cos\left(u(t)\right) \int_{-\infty}^{t-2} u(\tau)\mathrm{d}\tau.$$

A) Linear

C) Time-invariant

B) Causal

D) Static

FS 2022

Die Bewegungsgleichungen, basierend auf dem zweiten Newtonschen Gesetz ($F = M \cdot a$), lauten wie folgt:

$$
\begin{aligned}
k \cdot (x_2 - x_1) - c \cdot \dot{x}_1 &= m_1 \cdot \ddot{x}_1 \,, \\
F_\mathrm{u} - k \cdot (x_2 - x_1) &= m_2 \cdot \ddot{x}_2
\end{aligned}
$$

F9 (**1.5 Punkte**) Vervollständigen Sie die Zustandsraumdarstellung basierend auf den gegebenen Bewegungsgleichungen.

$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} = \left[\begin{array}{c|c|c|c} 0 & 1 & 0 & 0 \\ \hline & & & \\ \hline 0 & 0 & 0 & 1 \\ \hline & & & \end{array}\right] \cdot \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 \end{bmatrix} \cdot u(t)
$$

FS 2022

Die Bewegungsgleichungen, basierend auf dem zweiten Newtonschen Gesetz ($F = M \cdot a$), lauten wie folgt:

$$
\begin{aligned}
k \cdot (x_2 - x_1) - c \cdot \dot{x}_1 &= m_1 \cdot \ddot{x}_1 \,, \\
F_\mathrm{u} - k \cdot (x_2 - x_1) &= m_2 \cdot \ddot{x}_2
\end{aligned}
$$

F9 (**1.5 Punkte**) Vervollständigen Sie die Zustandsraumdarstellung basierend auf den gegebenen Bewegungsgleichungen.

$$
\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k/m_1 & -c/m_1 & k/m_1 & 0 \\ 0 & 0 & 0 & 1 \\ k/m_2 & 0 & -k/m_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 \end{bmatrix} \cdot u(t)
$$

LTI SISO

For LTI SISO Systems, we can represent out state space model in the following new form:

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t)).$$

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t),$$

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -\frac{k}{m}x_1(t) + \frac{1}{m}u(t).$$
$$y(t) = x_2(t).$$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + 0 \quad u(t).$$

Linearization

For non linear systems to be represented in the state space model, the **general procedure** looks like this:

- Find equilibrium point by solving $f(x_e, u_e) = 0.$
- Linearize around the equilibrium point using Jacobian-Linearization-Procedure, which is basically Taylors-series. More precisely, our matrices will look like this:

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{(x_e,u_e)} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{(x_e,u_e)}$$

$$B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{(x_e,u_e)} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \vdots \\ \frac{\partial f_n}{\partial u} \end{bmatrix}\right|_{(x_e,u_e)}$$

$$C = \left.\frac{\partial h(x,u)}{\partial x}\right|_{(x_e,u_e)} = \left.\begin{bmatrix} \frac{\partial h}{\partial x_1} & \cdots & \frac{\partial h}{\partial x_n} \end{bmatrix}\right|_{(x_e,u_e)}$$

$$D = \left.\frac{\partial h(x,u)}{\partial u}\right|_{(x_e,u_e)} = \left.\begin{bmatrix} \frac{\partial h}{\partial u} \end{bmatrix}\right|_{(x_e,u_e)}$$

Theory Recap

Time Response

Time Response

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t).$$

- Remember our general LTI SISO State space system, which analytically describes our system's behaviour.

- However, it does not yield an explicit input–output relation.
  For this, we will explicitly have to solve this system and obtain an expression y(t), basically the Σ from block diagrams.

- We also call y(t) the **time response**. And it tells us how the system responds to a specific initial condition and input over time

Compute Time Response

- First we need to find the expression for the state **x(t)**
  This we can then plug into the second equation for **y(t)**

$$\boxed{\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t).\end{aligned}}$$

- We use a similar approach as dividing an ODE into a homogenous and particular solution.
  We divide the response into initial condition response (**IC**) and forced response (**F**).

  This is possible, because we are looking at linear systems (**superposition**).
  (For more precise derivation, see lecture, or just take it as granted)

$$\boxed{y = y_{\mathrm{IC}} + y_{\mathrm{F}}}$$

$$\left.\begin{aligned} x_{\mathrm{IC}}(0) &= x_0,\\ u_{\mathrm{IC}}(t) &= 0, \quad t \geq t_0, \end{aligned}\right. \quad \rightarrow \quad y_{\mathrm{IC}}; \qquad + \qquad \left.\begin{aligned} x_{\mathrm{F}}(0) &= 0,\\ u_{\mathrm{F}}(t) &= u(t), \quad t \geq t_0, \end{aligned}\right. \quad \rightarrow \quad y_{\mathrm{F}}.$$

Interjection: Different Systems

We are gonna differentiate between the **LTI SISO** cases below.
First we we look at first order systems for simplicity, then we'll transition to the general case

First Order System:

$$\dot{x}(t) = ax(t) + bu(t),$$
$$y(t) = cx(t) + du(t).$$

a,b,c,d are scalars

n - Order System:

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t).$$

A,B,C,D are Matrices

Initial Condition Response

$$x_{\mathrm{IC}}(0) = x_0$$
$$u_{\mathrm{IC}}(t) = 0, \quad t \geq t_0,$$

- With our framework, our state space representations becomes:

$$\dot{x}(t) = ax(t) + bu(t), \qquad \dot{x}(t) = ax(t),$$
$$y(t) = cx(t) + du(t). \quad \longrightarrow \quad y(t) = cx(t).$$

- The solution to this system can be seen below. Remember this
- from Ana II / LinAlg II

$$x_{\mathrm{IC}}(t) = e^{at} x_0.$$

$$y_{\mathrm{IC}}(t) = c\, e^{at} x_0.$$

Forced Respone

$$x_{\mathrm{F}}(0) = 0,$$
$$u_{\mathrm{F}}(t) = u(t), \quad t \geq t_0,$$

- For the forced response it is not so easy anymore…

$$\dot{x}(t) = ax(t) + bu(t),$$
$$y(t) = cx(t) + du(t).$$

$\Rightarrow$

$$\dot{x}(t) = ax(t) + bu(t),$$
$$y(t) = cx(t) + du(t).$$

- The solution to this system can be seen below. See lecture for derivation.

$$x_{\mathrm{F}}(t) = \int_0^t e^{a(t-\tau)} bu(\tau) \; \mathrm{d}\,\tau.$$

$$y_{\mathrm{F}}(t) = c \int_0^t e^{a(t-\tau)} bu(\tau) \, \mathrm{d}\,\tau + du(t).$$

Complete Response First Order

- By combining the initial condition response and forced response, we get our complete response!

$$x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)} b u(\tau)\,\mathrm{d}\,\tau,$$

$$y(t) = \underbrace{c e^{at}x_0}_{\text{IC}} + c \underbrace{\int_0^t e^{a(t-\tau)} b u(\tau)\,\mathrm{d}\,\tau + d u(t)}_{\text{F}}.$$

- But wait, this was only for the simple first order systems! How does it change for n - order??

Complete Response for n - Order Systems

- The only thing that really changes are scalars. They now become matrices:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}\, B\, u(\tau)\,\mathrm{d}\tau,$$

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}\, B\, u(\tau)\,\mathrm{d}\tau + D\,u(t).$$

- But now how de we compute the terms with the Matrix A in the exponent? e^{At} and $e^{A(t-\tau)}$
LinAlg again!!

Similarity Transform

Similarity Transformation

$$e^{At} = \sum_{n=0}^{\infty} \frac{1}{n!}(At)^n = I + At + \frac{1}{2!}(At)^2 + \cdots + \frac{1}{n!}(At)^n.$$

- Computing the exponentials of a matrix is quite complicated.
  One way is to go via the taylor expansion. However there infinitely many terms…
  But there are matrices which drastically simplify the calculation:

- Diagonal Matrix:

$$\boxed{\exp\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} t\right) = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}}$$

- Jordan Form Matrix:

$$\boxed{\begin{gathered} \exp(\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} t) = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \\ \exp(\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} t) = e^{\lambda t} \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \end{gathered}}$$

Similarity Transformation

- But just as most physical systems are not linear (therefore we had to linearize the matrices), most of the matrices are not diagonal or in Jordan form. Therefore we need to find some transformation to get A into a desired form.
- For this we introduce an invertible matrix T for a coordinate transformation: $x = T\tilde{x}.$

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t) \end{aligned} \quad \Longrightarrow \quad \begin{aligned} T\dot{\tilde{x}}(t) &= AT\tilde{x}(t) + Bu(t), \\ y(t) &= CT\tilde{x}(t) + Du(t) \end{aligned}$$

$$\begin{aligned} \dot{\tilde{x}}(t) &= (T^{-1}AT)\tilde{x}(t) + (T^{-1}B)u(t), \\ y(t) &= CT\tilde{x}(t) + Du(t) \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \dot{\tilde{x}}(t) &= \tilde{A}\tilde{x}(t) + \tilde{B}u(t), \\ y(t) &= \tilde{C}\tilde{x}(t) + Du(t) \end{aligned}$$

Quizz

- When applying the similarity transformation, do we change the systems response??

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t)$$

$$\Rightarrow \Rightarrow \Rightarrow$$

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t),$$
$$y(t) = \tilde{C}\tilde{x}(t) + Du(t)$$

Diagonalization

- Okay, but why did we do this again?
  Remember from LinAlg, when a matrix $A \in R^{n \times n}$ has n linearly independent eigenvectors, it is called diagonalizable. In that case, we can find:

$$\tilde{A} = T^{-1}AT = diag(\lambda_1, \ldots, \lambda_n)$$

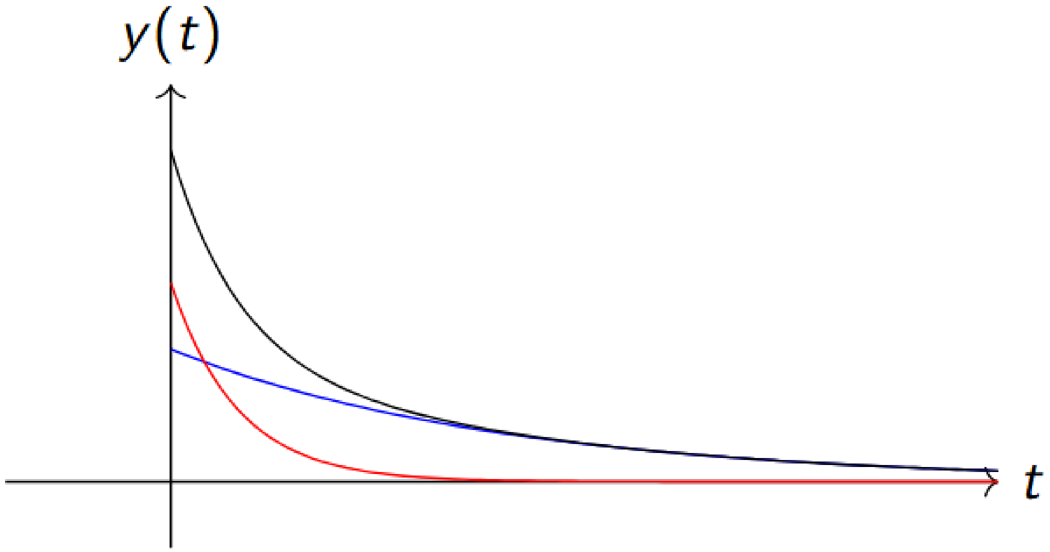
Remember? →

Matrix diagonalisieren (Basiswechsel in Eigenbasis)

1. Bestimme die Eigenwerte λ_i und die Eigenvektoren v_i
2. Die Matrix $D = diag(\lambda_1, \ldots, \lambda_n)$ ist eine Diagonalmatrix mit den Eigenwerten auf der Diagonalen.
3. Die Matrix $T = (v_1, \ldots, v_n)$ hat die Eigenvektoren als Spalten **(Gleiche Reihenfolge wie bei D!)**.
4. Bestimme T^{-1}. Falls EV orthonormal $T^{-1} = T^T$.

Example

$$y(t) = Ce^{At}x_0 = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix} = c_1 e^{\lambda_1 t} x_{0,1} + c_2 e^{\lambda_2 t} x_{0,2}$$

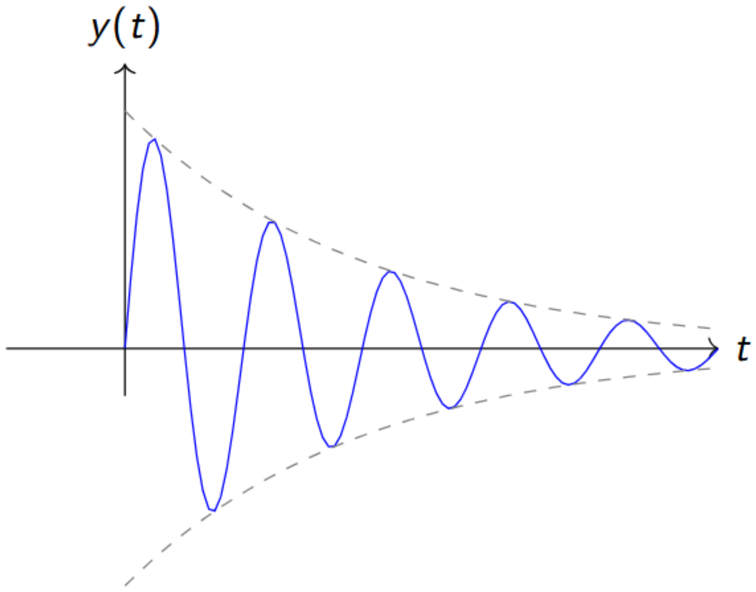


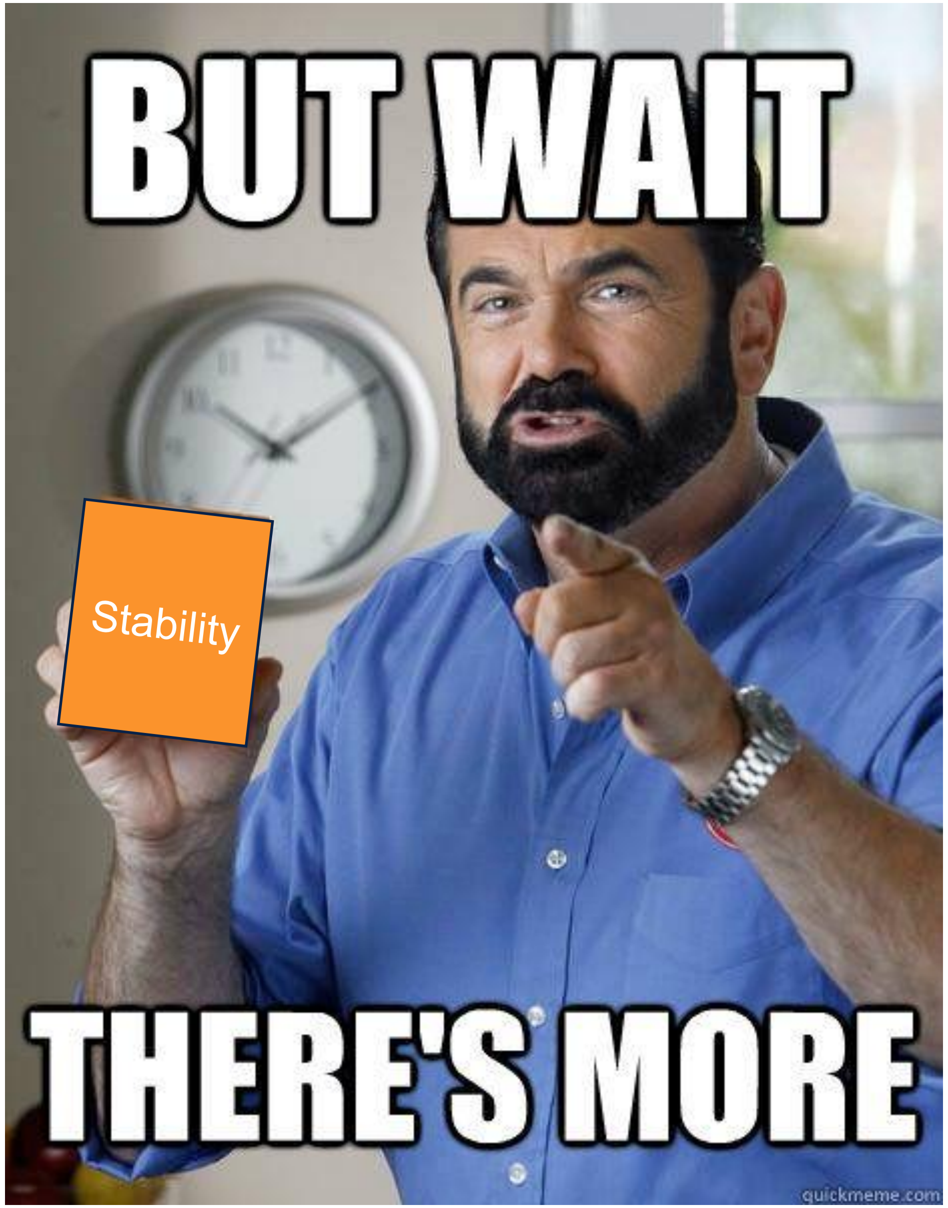
Example Complex Eigenvalues

$$y(t) = Ce^{At}x_0$$, A = $\begin{bmatrix} \sigma + j\omega & 0 \\ 0 & \sigma - j\omega \end{bmatrix}$

$$\begin{aligned} y(t) &= c_1 e^{\sigma t} e^{j\omega t} x_{0,1} + c_2 e^{\sigma t} e^{-j\omega t} x_{0,2} \\ &= e^{\sigma t}\left(c_1 e^{j\omega t} x_{0,1} + c_2 e^{-j\omega t} x_{0,2}\right) \\ &= e^{\sigma t}\left(\alpha_1 \sin(\omega t) + \alpha_2 \cos(\omega t)\right) \end{aligned}$$

$$y(t) = \alpha e^{\sigma t} \sin(\omega t + \varphi).$$




BUT WAIT
Stability
THERE'S MORE
quickmeme.com

Stability

Stability Classification

$$y(t) = \alpha e^{\sigma t} \sin(\omega t + \varphi).$$

Remember that the real part of our eigenvalues represents the exponential term of the response. Therefore, we can classify our stability by looking at them:

- Asymptotically Stable: State converges to zero for bounded initial conditions and zero input.

$$\mathrm{Re}(s) < 0 \text{ for all } \lambda_i.$$

- Lyapunov Stable: State will remain bounded for bounded initial conditions and zero input.

$$\mathrm{Re}(s) \leq 0 \text{ for all } \lambda_i.$$

- BIBO Stable: Output remains bounded for every bounded input.

- In Linear Systems: **Asymptotically stable → Lyapunov stable**
  Asymptotically stable → BIBO stable

HS 2022

Problem: You are given the following linear time-invariant system with state vector $x(t) \in \mathbb{R}^3$ and input $u(t) \in \mathbb{R}$:

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \cdot x(t)\,.$$

Q10 **(0.5 Points)** Mark the correct answer for each statement.

Statement	**True**	**False**
The system is asymptotically stable.		
The system is Lyapunov stable.		
The system is BIBO-stable.		

HS 2022

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot u(t)$$

$$\boxed{y_{IC}(t) = Ce^{At}x_0}$$

$$y(t) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \cdot x(t) \,.$$

(1 Points) Compute the initial condition response $y(t)$ with the initial condition $x(0) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$.

$$y(t) =$$

https://n.ethz.ch/~jschul/Feedback

Feedback

Q&A Session / Done